MATH 2050C Lecture 17 (Mar 16)

[Reminder: PS7 due today, PS8 due on Mar 19.]

Last week: limit of functions & sequential criteria

Setup: $f : A \subseteq \mathbb{R} \to \mathbb{R}$, c : a cluster pt. of A (Note: not nec. belong to A)

Def^{n}: $\lim_{x \to c} f(x) = L \iff \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $x \in A$ and $0 < |x - c| < \delta$

Sequential Criteria

$\lim_{x \to c} f(x) = L \iff \forall$ seq. (x_n) in A s.t. $\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$

we have $\lim (f(x_n)) = L$

limit of function

limit of seq. of real numbers

Remark: This is helpful, in particular, to show that the limit

$\lim_{x \to c} f(x)$ DOES NOT EXIST.

Taking the negation of Sequential Criteria above, we get:

Cor 1: f DOES NOT Converge to L as $x \to c$ $\iff \exists$ seq. (x_n) in A s.t. $\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$

BUT $(f(x_n)) \not\to L$

Cor 2: f "DIVERGES" as $x \to c$ $\iff \exists$ seq. (x_n) in A s.t. $\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$

BUT $(f(x_n))$ is divergent.

(i.e. f DOES NOT Converge to L $\forall L \in \mathbb{R}$ as $x \to c$)

"Divergence Criteria"

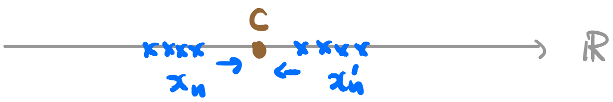
Proof of Cor. 2: "$\Leftarrow$" Easy.

"$\Rightarrow$" Argue by Contradiction. Assume f diverges at $x \to c$ but the R.H.S. fails to hold.

i.e. $\forall$ seq. (x_n) in A st. $(*) \begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim(x_n) = c \end{cases}$

we have $\lim(f(x_n)) = L$ for some $L \in \mathbb{R}$

which may depend on the sequence (x_n)



Claim: The limit L DOES NOT depend on (x_n).

Pf of Claim: Suppose (x_n), (x_n') satisfy $(*)$, and

$$\lim(f(x_n)) = L \neq L' = \lim(f(x_n'))$$

Consider the new sequence

$$(y_n) := (x_1, x_1', x_2, x_2', x_3, x_3', \ldots \ldots)$$

satisfies $(*)$, then by hypothesis

$$(f(y_n)) := (f(x_1), f(x_1'), f(x_2), f(x_2'), \ldots \ldots)$$

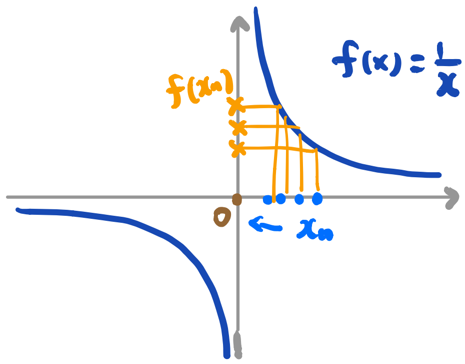
L, L'

is convergent, hence $L = L'$ ________ ▫

By sequential criteria, $\lim\limits_{x \to c} f(x) = L$ contradiction! ________ ▫

We now look at some examples where the limit of functions does not exist.

Example 1 : $\lim\limits_{x\to 0} \frac{1}{x}$ DOES NOT EXIST!



$f: A=\mathbb{R}\setminus\{0\} \to \mathbb{R}$

$f(x) := \frac{1}{x}$

Pf: Take $(x_n) := (\frac{1}{n})$.

Clearly, $\lim (x_n) = 0$, and

$A \ni x_n \neq 0 \quad \forall n \in \mathbb{N}$

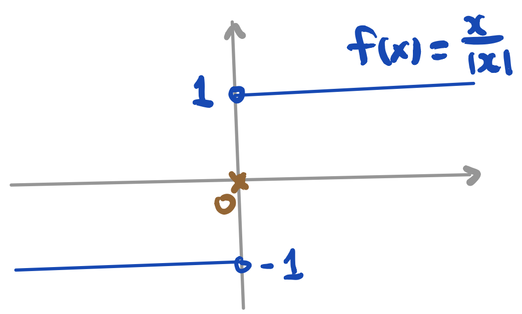
So $(*)$ is satisfied.

BUT $(f(x_n)) = (n)$ is DIVERGENT!

So we are done according to the divergence criteria above.

[Exercise: Prove directly using ε-δ def$^{\underline{n}}$ of limit.]

Example 2 : $\lim\limits_{x\to 0} \frac{x}{|x|}$ DOES NOT EXIST.



Pf: Take $(x_n) := \left(\frac{(-1)^n}{n}\right) \xrightarrow{*} 0$

satisfying $(*)$, the image seq.

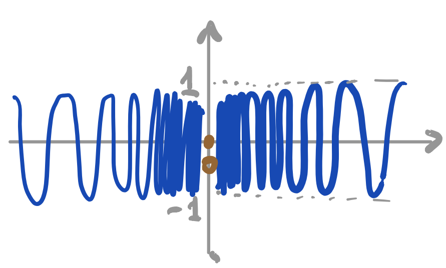
$(f(x_n)) = ((-1)^n)$ DIVERGENT!

$f: A=\mathbb{R}\setminus\{0\} \to \mathbb{R}$

$f(x) := \frac{x}{|x|}$

$\blacksquare$

Example 3 : $\lim\limits_{x\to 0} \sin\frac{1}{x}$ DOES NOT EXIST!



Pf: Take $(x_n) := \left(\frac{1}{n\pi}\right) \xrightarrow{*} 0$ BUT $(f(x_n)) = (0)$

Take $(x_n') := \left(\frac{1}{\frac{\pi}{2}+2n\pi}\right) \xrightarrow{*} 0$ BUT $(f(x_n')) = (1)$

So, let $(y_n) = (x_1, x_1', x_2, x_2', x_3, x_3', \ldots) \xrightarrow{*} 0$

BUT $(f(y_n)) = (0, 1, 0, 1, 0, 1, \ldots)$ DIVERGENT!

$f: A=\mathbb{R}\setminus\{0\} \to \mathbb{R}$

$f(x) = \sin\frac{1}{x}$

$\blacksquare$

Limit Theorems for functions (§ 4.2 in textbook)

(iff statement)

Motto: By Sequential Criteria, we get limit theorems for functions from the corresponding limit theorems for sequences.

Recall: For sequences, (x_n) convergent $\Rightarrow$ (x_n) bdd

We have a similar result for functions.

Boundedness Thm:

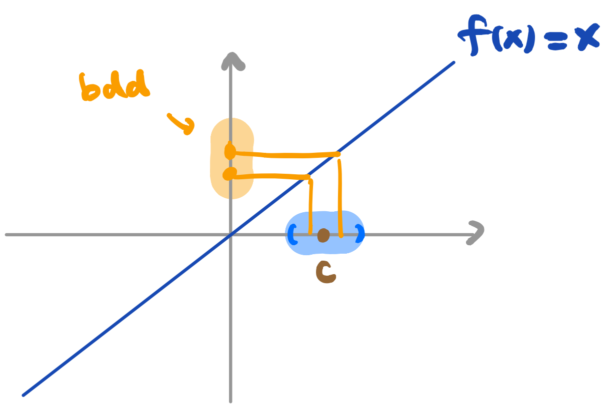
$\lim_{x \to c} f(x)$ exists $\Rightarrow$ f is "bdd in a neighborhood of c"

i.e. $\exists\, M > 0$ and $\exists\, \delta > 0$ s.t.

(Note: $\Leftarrow$ not true) $\quad |f(x)| \le M \quad \forall\, |x - c| < \delta$ and $x \in A$

Remark: f may not be bdd "globally".

E.g.)



Proof: By ε-δ def^{n}, $\lim_{x \to c} f(x) = L$ $\Rightarrow$ Take $\varepsilon = 1$

Then $\exists\, \delta = \delta(1) > 0$ s.t. $|f(x) - L| < \varepsilon = 1$

whenever $x \in A$ and $0 < |x - c| < \delta$

by Δ-ineq

$\Rightarrow \begin{cases} |f(x)| \le |f(x) - L| + |L| < 1 + |L| \\ \text{whenever } x \in A \text{ and } 0 < |x - c| < \delta \end{cases}$

If we take $M := \max\{1+|L|, \underbrace{|f(c)|}_{\text{if } c\in A}\} > 0$, then

we have $|f(x)| \le M \quad \forall x \in A$ s.t $|x-c| < \delta$

$\blacksquare$

<u>Def^{n}</u>: Given $f, g : A \to \mathbb{R}$ functions defined on the same A,

then we can define new functions:

- $(f \pm g)(x) := f(x) \pm g(x) \qquad f\pm g : A \to \mathbb{R}$
- $(fg)(x) := f(x)g(x) \qquad fg : A \to \mathbb{R}$
- $\left(\frac{f}{g}\right)(x) := \dfrac{f(x)}{g(x)} \qquad \frac{f}{g} : A \setminus \{x\in A \mid g(x)=0\} \to \mathbb{R}$

<u>Thm</u>: (1) $\lim\limits_{x\to c} (f\pm g)(x) = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$

(2) $\lim\limits_{x\to c} (fg)(x) = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$

**(3) $\lim\limits_{x\to c} \left(\frac{f}{g}\right)(x) = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$

extra careful!

provided that $\lim\limits_{x\to c} f(x)$, $\lim\limits_{x\to c} g(x)$ exist,

and for (3), additionally, we need $\lim\limits_{c\to c} g(x) \neq 0$

<u>Examples</u>: $\lim\limits_{x\to c} \frac{1}{x} = \frac{1}{c}$ provided $c \neq 0$; $\lim\limits_{x\to 3} \dfrac{x^3-4}{x+1} = \dfrac{4}{3}$

$\lim\limits_{x\to 2} \dfrac{x^2-4}{3x-6} = \lim\limits_{x\to 2} \dfrac{(x+2)\cancel{(x-2)}}{3\cancel{(x-2)}} = \dfrac{4}{3}$.

Proof of (2): IDEA: Use seq criteria.

Take (x_n) in A s.t. $x_n \neq c$ $\forall n \in \mathbb{N}$ and $\lim (x_n) = c$.

Seq criteria $\Rightarrow$ $(f(x_n)) \to \lim\limits_{x \to c} f(x)$ & $(g(x_n)) \to \lim\limits_{x \to c} g(x)$

Limit Thm for seq. $\Rightarrow$ $(f(x_n) \cdot g(x_n)) \to \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x)$

Seq criteria $\Rightarrow$ $\lim\limits_{x \to c} (fg)(x) = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x)$

$\square$